Summer Review Packet AB Calculus /High School Calculus Supplemental Materials Packet

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These materials aren't in the exact same order as what the homework pages are in, and it doesn't cover quite everything in the packet, but I think it's still worth sending to you.

I'll list the topics out below. Hopefully, that will enable you to find review material more quickly when you need it. You can also look topics up online or email me (beth.hill@icregina.com).

A. Functions
B. Domain and Range
C. Graphs of Common Functions (*Please familiarize yourself with these)
D. Even and Odd Functions
E. Transformations of Graphs
F. Special Factorization
G. Linear Functions
H. Solving Quadratic Equations
I. Asymptotes
J. Negative and Fractional Exponents
K. Eliminating Complex Fractions
L. Inverses
M. Adding Fractions and Solving Fractional Equations
N. Solving Absolute Value Equations
O. Solving Inequalities
P. Exponential Functions and Logarithms
Q. Right Angle Trigonometry
R. Special Angles
S. Trigonometric Identities
T. Solving Trig Equations and Inequalities
U. Graphical Solutions to Equations and Inequalities

I look forward to seeing everyone in August!

Mrs. Hill

A. Functions

The lifeblood of precalculus is functions. A **function** is a set of points (x,y) such that for every x, there is one and only one y. In short, in a function, the x-values cannot repeat while the y-values can. In AB Calculus, all of your graphs will come from functions.

The notation for functions is either "$y=$" or "$f(x)=$". In the $f(x)$ notation, we are stating a rule to find y given a value of x.

1. If $f(x)=x^2-5x+8$, find a) $f(-6)$ b) $f\left(\frac{3}{2}\right)$ c) $\frac{f(x+h)-f(x)}{h}$

a) $f(-6)=(-6)^2-5(-6)+8$

$36+30+8$

74

b) $f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^2-5\left(\frac{3}{2}\right)+8$

$\frac{9}{4}-\frac{15}{2}+8$

$\frac{11}{4}$

c) $\frac{f(x+h)-f(x)}{h}$

$\frac{(x+h)^2-5(x+h)+8-(x^2-5x+8)}{h}$

$\frac{x^2+2xh+h^2-5x-5h+8-x^2+5x-8}{5}$

$\frac{h^2+2xh-5h}{h}=\frac{h(h+2x-5)}{h}=h+2x-5$

Functions do not always use the variable x. In calculus, other variables are used liberally.

2. If $A(r)=\pi r^2$, find a) $A(3)$ b) $A(2s)$ c) $A(r+1)-A(r)$

$\boxed{A(3)=9\pi}$

$\boxed{A(2s)=\pi(2s)^2=4\pi s^2}$

$A(r+1)-A(r)=\pi(r+1)^2-\pi r^2$

$\pi(2r+1)$

One concept that comes up in AP calculus is **composition of functions**. The format of a composition of functions is: plug a value into one function, determine an answer, and plug that answer into a second function.

3. If $f(x)=x^2-x+1$ and $g(x)=2x-1$, a) find $f(g(-1))$ b) find $g(f(-1))$ c) show that $f(g(x))\neq g(f(x))$

$g(-1)=2(-1)-1=-3$

$f(-3)=9+3+1=13$

$f(-1)=1+1+1=3$

$g(3)=2(3)-1=5$

$f(g(x))=f(2x-1)=(2x-1)^2-(2x-1)+1$

$=4x^2-4x+1-2x+1+1=4x^2-6x+3$

$g(f(x))=g(x^2-x+1)=2(x^2-x+1)-1$

$=2x^2-2x+1$

Finally, expect to use **piecewise functions**. A piecewise function gives different rules, based on the value of x.

4. If $f(x)=\begin{cases} x^2-3, & x\geq 0 \\ 2x+1, & x<0 \end{cases}$, find a) $f(5)$ b) $f(2)-f(-1)$ c) $f(f(1))$

$\boxed{f(5)=25-3=22}$

$\boxed{f(2)-f(-1)=1-(-1)=2}$

$\boxed{f(1)=-2,\ f(-2)=-3}$

B. Domain and Range

First, since questions in calculus usually ask about behavior of functions in intervals, understand that intervals can be written with a description in terms of $<, \leq, >, \geq$ or by using **interval notation**.

Description	Interval notation	Description	Interval notation	Description	Interval notation
$x > a$	(a,∞)	$x \leq a$	$(-\infty,a]$	$a \leq x < b$	$[a,b)$
$x \geq a$	$[a,\infty)$	$a < x < b$	(a,b) - open interval	$a < x \leq b$	$(a,b]$
$x < a$	$(-\infty,a)$	$a \leq x \leq b$	$[a,b]$ - closed interval	All real numbers	$(-\infty,\infty)$

If a solution is in one interval or the other, interval notation will use the connector $\cup$. So $x \leq 2$ or $x > 6$ would be written $(-\infty,2]\cup(6,\infty)$ in interval notation. Solutions in intervals are usually written in the easiest way to define it. For instance, saying that $x < 0$ or $x > 0$ or $(-\infty,0)\cup(0,\infty)$ is best expressed as $x \neq 0$.

The **domain of a function** is the set of allowable x-values. The domain of a function f is $(-\infty,\infty)$ except for values of x which create a zero in the denominator, an even root of a negative number or a logarithm of a non-positive number. The domain of $a^{p(x)}$ where a is a positive constant and $p(x)$ is a polynomial is $(-\infty,\infty)$.

• Find the domain of the following functions using interval notation:

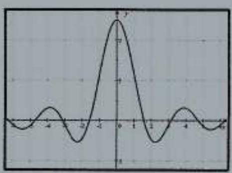
1. $f(x)=x^2-4x+4$ $\boxed{(-\infty,\infty)}$
2. $y=\dfrac{6}{x-6}$ $\boxed{x \neq 6}$
3. $y=\dfrac{2x}{x^2-2x-3}$ $\boxed{x \neq -1, x \neq 3}$
4. $y=\sqrt{x+5}$ $\boxed{[-5,\infty)}$
5. $y=\sqrt[3]{x+5}$ $\boxed{(-\infty,\infty)}$
6. $y=\dfrac{x^2+4x+6}{\sqrt{2x+4}}$ $\boxed{(-2,\infty)}$

The **range of a function** is the set of allowable y-values. Finding the range of functions algebraically isn't as easy (it really is a calculus problem), but visually, it is the [lowest possible y-value, highest possible y-value]. Finding the range of some functions are fairly simple to find if you realize that the range of $y=x^2$ is $[0,\infty)$ as any positive number squared is positive. Also the range of $y=\sqrt{x}$ is also positive as the domain is $[0,\infty)$ and the square root of any positive number is positive. The range of $y=a^x$ where a is a positive constant is $(0,\infty)$ as constants to powers must be positive.

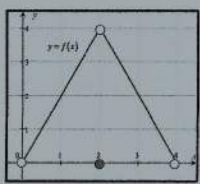
• Find the range of the following functions using interval notation:

7. $y=1-x^2$ $\boxed{(-\infty,1]}$
8. $y=\dfrac{1}{x^2}$ $\boxed{(0,\infty)}$
9. $y=\sqrt{x-8}+2$ $\boxed{[2,\infty)}$

• Find the domain and range of the following functions using interval notation.

10. 

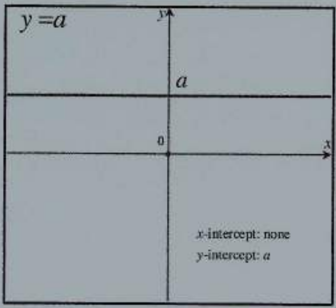
Domain: $(-\infty,\infty)$
Range: $[-0.5,2.5]$

11. 

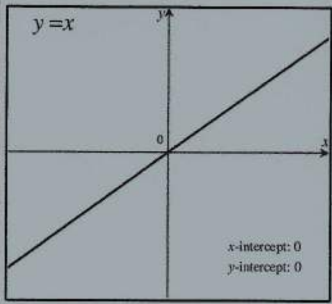
Domain: $(0,4)$
Range: $[0,4)$

C. Graphs of Common Functions

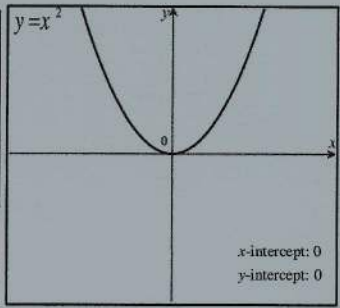
There are certain graphs that occur all the time in calculus and students should know the general shape of them, where they hit the x-axis (zeros) and y-axis (y-intercept), as well as the domain and range. There are no assignment problems for this section other than students memorizing the shape of all of these functions. In section 5, we will talk about transforming these graphs.



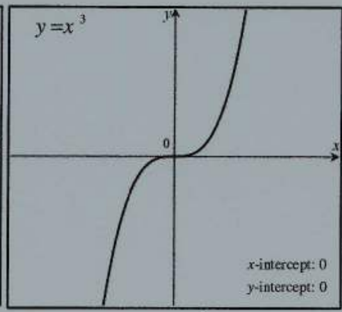
Function: $y = a$
Domain: $(-\infty, \infty)$
Range: $[a, a]$



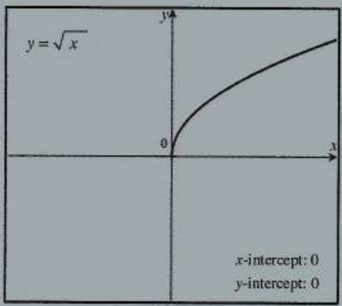
Function: $y = x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



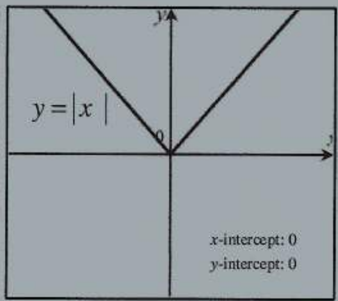
Function: $y = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



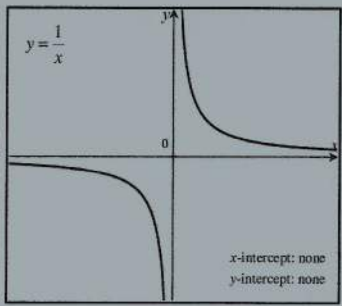
Function: $y = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



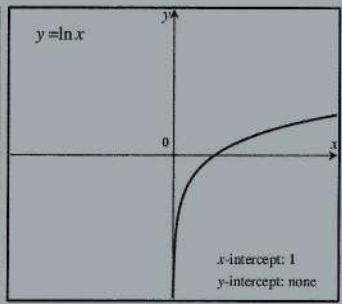
Function: $y = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$



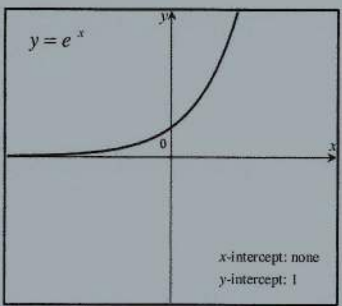
Function: $y = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



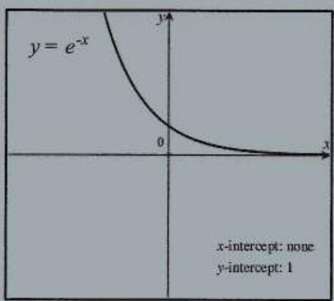
Function: $y = \dfrac{1}{x}$
Domain: $x \neq 0$
Range: $y \neq 0$



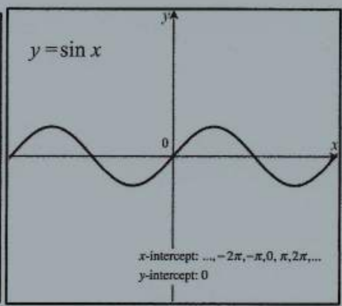
Function: $y = \ln x$
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$



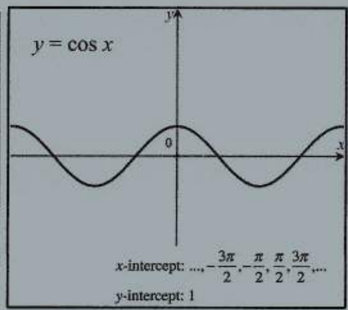
Function: $y = e^x$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$



Function: $y = e^{-x}$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$



Function: $y = \sin x$
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



Function: $y = \cos x$
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$

D. Even and Odd Functions

Functions that are even have the characteristic that for all a, $f(-a)=f(a)$. What this says is that plugging in a positive number a into the function or a negative number $-a$ into the function makes no difference … you will get the same result. Even functions are symmetric to the y-axis.

Functions that are odd have the characteristic that for all a, $f(-a)=-f(a)$. What this says is that plugging in a negative number $-a$ into the function will give you the same result as plugging in the positive number and taking the negative of that. So, odd functions are symmetric to the origin. If a graph is symmetric to the x-axis, it is not a function because it fails the vertical-line test.

1. Of the common functions in section 3, which are even, which are odd, and which are neither?

Even: $y=a$, $y=x^2$, $y=|x|$, $y=\cos x$ Odd: $y=x$, $y=x^3$, $y=\frac{1}{x}$, $y=\sin x$

Neither: $y=\sqrt{x}$, $y=\ln x$, $y=e^x$, $y=e^{-x}$

2. Show that the following functions are even:

a) $f(x)=x^4-x^2+1$

$f(-x)=(-x)^4-(-x)^2+1$
$=x^4-x^2+1=f(x)$

b) $f(x)=\left|\frac{1}{x}\right|$

$f(-x)=\left|\frac{1}{-x}\right|=\left|\frac{1}{x}\right|=f(x)$

c) $f(x)=x^{2/3}$

$f(-x)=(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2$
$=\left(\sqrt[3]{x}\right)^2=f(x)$

3. Show that the following functions are odd:

a) $f(x)=x^3-x$

$f(-x)=(-x)^3+x$
$=x-x^3=-f(x)$

b) $f(x)=\sqrt[3]{x}$

$f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)$

c) $f(x)=e^x-e^{-x}$

$f(-x)=e^{-x}-e^x=-\left(e^x-e^{-x}\right)=-f(x)$

4. Determine if $f(x)=x^3-x^2+x-1$ is even, odd, or neither. Justify your answer.

$f(-x)=-x^3-x^2-x-1\neq f(x)$ so f is not even. $-f(x)=-x^3+x^2-x-1\neq f(-x)$ so f is not odd.

Graphs may not be functions and yet have x-axis or y-axis or both. Equations for these graphs are usually expressed in "implicit form" where it is not expressed as "$y=$" or "$f(x)=$". If the equation does not change after making the following replacements, the graph has these symmetries:

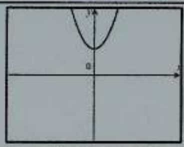
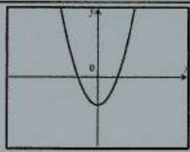
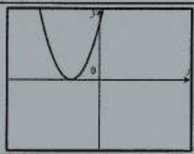
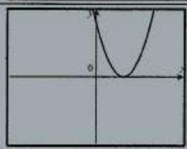
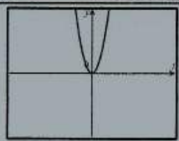
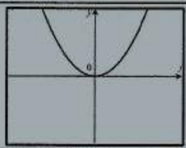
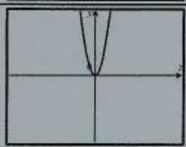
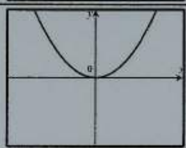
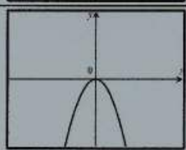
x-axis: y with $-y$ y-axis: x with $-x$ origin: both x with $-x$ and y with $-y$

5. Determine the symmetry for $x^2+xy+y^2=0$.

$x-$axis: $x^2+x(-y)+(-y)^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $x-$axis
$y-$axis: $(-x)^2+(-x)(y)+y^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $y-$axis
origin: $(-x)^2+(-x)(-y)+y^2=0\Rightarrow x^2+xy+y^2=0$ so symmetric to origin

E. Transformation of Graphs

A curve in the form $y = f(x)$, which is one of the basic common functions from section C can be transformed in a variety of ways. The shape of the resulting curve stays the same but zeros and y-intercepts might change and the graph could be reversed. The table below describes transformations to a general function $y = f(x)$ with the parabolic function $f(x) = x^2$ as an example.

Notation	How $f(x)$ changes	Example with $f(x) = x^2$
$f(x)+a$	Moves graph up a units	
$f(x)-a$	Moves graph down a units	
$f(x+a)$	Moves graph a units left	
$f(x-a)$	Moves graph a units right	
$a \cdot f(x)$	$a > 1$: Vertical Stretch	
$a \cdot f(x)$	$0 < a < 1$: Vertical shrink	
$f(ax)$	$a > 1$: Horizontal compress (same effect as vertical stretch)	
$f(ax)$	$0 < a < 1$: Horizontal elongated (same effect as vertical shrink)	
$-f(x)$	Reflection across x-axis	
$f(-x)$	Reflection across y-axis	

F. Special Factorization

While factoring skills were more important in the days when A topics were specifically tested, students still must know how to factor. The special forms that occur most regularly are:

Common factor: $x^3+x^2+x=x\left(x^2+x+1\right)$

Difference of squares: $x^2-y^2=(x+y)(x-y)$ or $x^{2n}-y^{2n}=\left(x^n+y^n\right)\left(x^n-y^n\right)$

Perfect squares: $x^2+2xy+y^2=(x+y)^2$

Perfect squares: $x^2-2xy+y^2=(x-y)^2$

Sum of cubes: $x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$ - Trinomial unfactorable

Difference of cubes: $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$ - Trinomial unfactorable

Grouping: $xy+xb+ay+ab=x(y+b)+a(y+b)=(x+a)(y+b)$

The term "factoring" usually means that coefficients are rational numbers. For instance, x^2-2 could technically be factored as $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$ but since $\sqrt{2}$ is not rational, we say that x^2-2 is not factorable. It is important to know that x^2+y^2 is unfactorable.

• Completely factor the following expressions.

1. $4a^2+2a$

   $\boxed{2a(2a+1)}$

2. $x^2+16x+64$

   $\boxed{(x+8)^2}$

3. $4x^2-64$

   $\boxed{4(x+4)(x-4)}$

4. $5x^4-5y^4$

   $\boxed{5\left(x^2+y^2\right)(x+y)(x-y)}$

5. $16x^2-8x+1$

   $\boxed{(4x-1)^2}$

6. $9a^4-a^2b^2$

   $\boxed{a^2(3a+b)(3a-b)}$

7. $2x^2-40x+200$

   $\boxed{2(x-10)^2}$

8. x^3-8

   $\boxed{(x-2)\left(x^2+2x+4\right)}$

9. $8x^3+27y^3$

   $\boxed{(2x+3y)\left(4x^2-6xy+9y^2\right)}$

10. x^4+11x^2-80

    $\boxed{(x+4)(x-4)\left(x^2+5\right)}$

11. x^4-10x^2+9

    $\boxed{(x+1)(x-1)(x+3)(x-3)}$

12. $36x^2-64$

    $\boxed{4(3x+4)(3x-4)}$

13. x^3-x^2+3x-3

    $x^2(x-1)+3(x-1)$

    $(x-1)\left(x^2+3\right)$

14. $x^3+5x^2-4x-20$

    $x^2(x+5)-4(x+5)$

    $(x+5)(x-2)(x+2)$

15. $9-\left(x^2+2xy+y^2\right)$

    $9-(x+y)^2$

    $(3+x+y)(3-x-y)$

G. Linear Functions

Probably the most important concept from precalculus that is required for differential calculus is that of linear functions. The formulas you need to know backwards and forwards are:

Slope: Given two points (x_1, y_1) and (x_2, y_2), the slope of the line passing through the points can be written as: $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Slope intercept form: the equation of a line with slope m and y-intercept b is given by $y = mx + b$.

Point-slope form: the equation of a line passing through the points (x_1, y_1) and slope m is given by $y - y_1 = m(x - x_1)$. While you might have preferred the simplicity of the $y = mx + b$ form in your algebra course, the $y - y_1 = m(x - x_1)$ form is far more useful in calculus.

Intercept form: the equation of a line with x-intercept a and y-intercept b is given by $\frac{x}{a} + \frac{y}{b} = 1$.

General form: $Ax + By + C = 0$ where A, B and C are integers. While your algebra teacher might have required your changing the equation $y - 1 = 2(x - 5)$ to general form $2x - y - 9 = 0$, you will find that on the AP calculus test, it is sufficient to leave equations for a lines in point-slope form and it is recommended not to waste time changing it unless you are specifically told to do so.

Parallel lines Two distinct lines are parallel if they have the same slope: $m_1 = m_2$.

Normal lines: Two lines are normal (perpendicular) if their slopes are negative reciprocals: $m_1 \cdot m_2 = -1$.

Horizontal lines have slope zero. **Vertical lines** have no slope (slope is undefined).

1. Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m = -4, (1,2)$

$\boxed{y - 2 = -4(x - 1) \Rightarrow y = -4x + 6}$

b. $m = \frac{2}{3}, (-5,1)$

$\boxed{y - 1 = \frac{2}{3}(x - 5) \Rightarrow y = \frac{2x}{3} - \frac{7}{3}}$

c. $m = 0, \left(-\frac{1}{2}, \frac{3}{4}\right)$

$\boxed{y = -\frac{3}{4}}$

2. Find the equation of the line in slope-intercept form, passing through the following points.

a. $(4,5)$ and $(-2,-1)$

$$m = \frac{5+1}{4+2} = 1$$
$$y - 5 = x - 4 \Rightarrow y = x + 1$$

b. $(0,-3)$ and $(-5,3)$

$$m = \frac{3+3}{-5-0} = \frac{-6}{5}$$
$$y + 3 = \frac{-6}{5}x \Rightarrow y = \frac{-6}{5}x - 3$$

c. $\left(\frac{3}{4}, -1\right)$ and $\left(1, \frac{1}{2}\right)$

$$m = \left(\frac{\frac{1}{2}+1}{1-\frac{3}{4}}\right)\left(\frac{4}{4}\right) = \frac{2+4}{4-3} = 6$$
$$y - \frac{1}{2} = 6(x - 1) \Rightarrow y = 6x - \frac{11}{2}$$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(4,7)$, $4x - 2y = 1$

$$y = 2x - \frac{1}{2} \Rightarrow m = 2$$
a) $y - 7 = 2(x - 4)$ b) $y - 7 = \frac{-1}{2}(x - 4)$

b. $\left(\frac{2}{3}, 1\right)$, $x + 5y = 2$

$$y = \frac{-1}{5}x + 2 \Rightarrow m = \frac{-1}{5}$$
a) $y - 1 = \frac{-1}{5}\left(x - \frac{2}{3}\right)$ b) $y - 1 = 5\left(x - \frac{2}{3}\right)$

H. Solving Quadratic Equations

Solving quadratics in the form of $ax^2+bx+c=0$ usually show up on the AP exam in the form of expressions that can easily be factored. But occasionally, you will be required to use the quadratic formula. When you have a quadratic equation, factor it, set each factor equal to zero and solve. If the quadratic equation doesn't factor or if factoring is too time-consuming, use the quadratic formula:

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. The **discriminant** b^2-4ac will tell you how many solutions the quadratic has:

$$b^2-4ac\begin{cases}>0,\ 2\text{ real solutions (if a perfect square, the solutions are rational)}\\=0,\ 1\text{ real solution}\\<0,\ 0\text{ real solutions (or 2 imaginary solutions, but AP calculus does not deal with imaginaries)}\end{cases}$$

1. Solve for x.

a. $x^2+3x+2=0$

$(x+2)(x+1)=0$
$x=-2, x=-1$

b. $x^2-10x+25=0$

$(x-5)^2=0$
$x=5$

c. $x^2-64=0$

$(x-8)(x+8)=0$
$x=8, x=-8$

d. $2x^2+9x=18$

$(2x-3)(x+6)=0$
$x=\dfrac{3}{2}, x=-6$

e. $12x^2+23x=-10$

$(4x+5)(3x+2)=0$
$x=-\dfrac{5}{4}, x=-\dfrac{2}{3}$

f. $48x-64x^2=9$

$(8x-3)^2=0$
$x=\dfrac{3}{8}$

g. $x^2+5x=2$

$x=\dfrac{-5\pm\sqrt{25+8}}{2}$
$x=\dfrac{-5\pm\sqrt{33}}{2}$

h. $8x-3x^2=2$

$x=\dfrac{8\pm\sqrt{64-24}}{6}$
$x=\dfrac{8\pm2\sqrt{10}}{6}=\dfrac{4\pm\sqrt{10}}{3}$

i. $6x^2+5x+3=0$

$x=\dfrac{-5\pm\sqrt{25-72}}{12}=\dfrac{-5\pm\sqrt{-47}}{12}$
No real solutions

j. $x^3-3x^2+3x-9=0$

$x^2(x-3)-3(x-3)=0$
$(x-3)(x^2-3)=0$
$x=3,\ x=\pm\sqrt{3}$

k. $\dfrac{x}{3}-\dfrac{5}{2}=\dfrac{-3}{x}$

$6x\left(\dfrac{x}{3}-\dfrac{5}{2}=\dfrac{-3}{x}\right)$
$2x^2-15x+18=0$
$(2x-3)(x-6)=0$
$x=\dfrac{3}{2},\ x=6$

l. $x^4-7x^2-8=0$

$(x^2-8)(x^2+1)=0$
$x=\pm\sqrt{8}=\pm2\sqrt{2}$

2. If $y=5x^2-3x+k$, for what values of k will the quadratic have two real solutions?

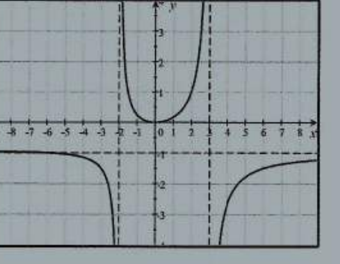
$$(-3)^2-4(5)k>0\Rightarrow 9-20k>0\Rightarrow k<\frac{9}{20}$$

I. Asymptotes

Rational functions in the form of $y=\dfrac{p(x)}{q(x)}$ possibly have vertical asymptotes, lines that the graph of the curve approach but never cross. To find the **vertical asymptotes**, factor out any common factors of numerator and denominator, reduce if possible, and then set the denominator equal to zero and solve.

Horizontal asymptotes are lines that the graph of the function approaches when x gets very large or very small. While you learn how to find these in calculus, a rule of thumb is that if the highest power of x is in the denominator, the horizontal asymptote is the line $y = 0$. If the highest power of x is both in numerator and denominator, the horizontal asymptote will be the line $y=\dfrac{\text{highest degree coefficient in numerator}}{\text{highest degree coefficient in denominator}}$. If the highest power of x is in the numerator, there is no horizontal asymptote, but a slant asymptote which is not used in calculus.

1) Find any vertical and horizontal asymptotes for the graph of $y=\dfrac{-x^2}{x^2-x-6}$.

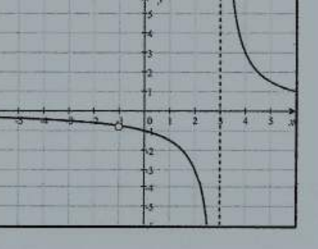


$$y=\frac{-x^2}{x^2-x-6}=\frac{-x^2}{(x-3)(x+2)}$$

Vertical asymptotes: $x-3=0 \Rightarrow x=3$ and $x+2=0 \Rightarrow x=-2$
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y=-1$.

This is confirmed by the graph to the right. Note that the curve actually crosses its horizontal asymptote on the left side of the graph.

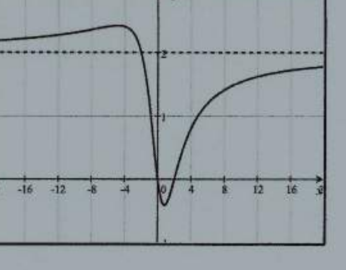
2) Find any vertical and horizontal asymptotes for the graph of $y=\dfrac{3x+3}{x^2-2x-3}$.



$$y=\frac{3x+3}{x^2-2x-3}=\frac{3(x+1)}{(x-3)(x+1)}=\frac{3}{x-3}$$

Vertical asymptotes: $x-3=0 \Rightarrow x=3$. Note that since the $(x+1)$ cancels, there is no vertical asymptote at $x = 1$, but a hole (sometimes called a removable discontinuity) in the graph.
Horizontal asymptotes: Since there the highest power of x is in the denominator, there is a horizontal asymptote at $y = 0$ (the x-axis). This is confirmed by the graph to the right.

3) Find any vertical and horizontal asymptotes for the graph of $y=\dfrac{2x^2-4x}{x^2+4}$.



$$y=\frac{2x^2-4x}{x^2+4}=\frac{2x(x-2)}{x^2+4}$$

Vertical asymptotes: None. The denominator doesn't factor and setting it equal to zero has no solutions.
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y = 2$. This is confirmed by the graph to the right.

J. Negative and Fractional Exponents

In calculus, you will be required to perform algebraic manipulations with **negative exponents** as well as **fractional exponents**. You should know the definition of a negative exponent: $x^{-n} = \frac{1}{x^n}, x \neq 0$. Note that negative powers do not make expressions negative; they create fractions. Typically expressions in multiple-choice answers are written with positive exponents and students are required to eliminate negative exponents. Fractional exponents create roots. The definition of $x^{1/2} = \sqrt{x}$ and $x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$.

As a reminder: when we multiply, we add exponents: $\left(x^a\right)\left(x^b\right) = x^{a+b}$.

When we divide, we subtract exponents: $\frac{x^a}{x^b} = x^{a-b}, x \neq 0$

When we raise powers, we multiply exponents: $\left(x^a\right)^b = x^{ab}$

In your algebra course, leaving an answer with a radical in the denominator was probably not allowed. You had to rationalize the denominator: $\frac{1}{\sqrt{x}}$ changed to $\left(\frac{1}{\sqrt{x}}\right)\left(\frac{\sqrt{x}}{\sqrt{x}}\right) = \frac{\sqrt{x}}{x}$. In calculus, you will find that it is not necessary to rationalize and it is recommended that you not take the time to do so.

• Simplify and write with positive exponents. Note: # 12 involves complex fractions, covered in section K.

1. $-8x^{-2}$

$$\boxed{\frac{-8}{x^2}}$$

2. $\left(-5x^3\right)^{-2}$

$$\boxed{(-5)^{-2}x^{-6} = \frac{1}{(-5)^2 x^6} = \frac{1}{25x^6}}$$

3. $\left(\frac{-3}{x^4}\right)^{-2}$

$$\boxed{\frac{(-3)^{-2}}{\left(x^4\right)^{-2}} = \frac{1}{(-3)^2 x^{-8}} = \frac{x^8}{9}}$$

4. $\left(36x^{10}\right)^{1/2}$

$$\boxed{6x^5}$$

5. $\left(27x^3\right)^{-2/3}$

$$\boxed{\frac{1}{\left(27x^3\right)^{2/3}} = \frac{1}{9x^2}}$$

6. $\left(16x^{-2}\right)^{3/4}$

$$\boxed{16^{3/4}x^{-4/3} = \frac{8}{x^{4/3}}}$$

7. $\left(x^{1/2} - x\right)^{-2}$

$$\boxed{\frac{1}{\left(x^{1/2} - x\right)^2} = \frac{1}{x - 2x^{3/2} + x^2}}$$

8. $\left(4x^2 - 12x + 9\right)^{-1/2}$

$$\boxed{\frac{1}{\left[(2x-3)^2\right]^{1/2}} = \frac{1}{2x-3}}$$

9. $\left(x^{1/3}\right)\left(\frac{1}{2}x^{-1/2}\right) + \left(x^{1/2} + 1\right)\left(\frac{1}{3}x^{-1/3}\right)$

$$\boxed{\frac{x^{1/3}}{2x^{1/2}} + \frac{x^{1/2}+1}{3x^{1/3}} = \frac{1}{2x^{1/6}} + \frac{x^{1/2}+1}{3x^{1/3}}}$$

10. $\frac{-2}{3}(8x)^{-5/3}(8)$

$$\boxed{\frac{-16}{3(8x)^{5/3}} = \frac{-16}{3(32)x^{5/3}} = -\frac{1}{6x^{5/3}}}$$

11. $\frac{(x+4)^{1/2}}{(x-4)^{-1/2}}$

$$\boxed{(x+4)^{1/2}(x-4)^{1/2} = \left(x^2 - 16\right)^{1/2}}$$

12. $\left(x^{-1} + y^{-1}\right)^{-1}$

$$\boxed{\left(\frac{1}{\frac{1}{x} + \frac{1}{y}}\right)\left(\frac{xy}{xy}\right) = \frac{xy}{y+x}}$$

K. Eliminating Complex Fractions

Calculus frequently uses **complex fractions**, which are fractions within fractions. Answers are never left with complex fractions and they must be eliminated. There are two methods to eliminate complex fractions:

When the problem is in the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, we can "flip the denominator" and write it as $\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$.
However, this does not work when the numerator and denominator are not single fractions. The best way is to eliminate the complex fractions in all cases is to find the LCD (lowest common denominator) of all the fractions in the complex fraction. Multiply all terms by this LCD and you are left with a fraction that is magically no longer complex. **Important:** Note that $\frac{x^{-1}}{y^{-1}}$ can be written as $\frac{y}{x}$ but $\frac{1+x^{-1}}{y^{-1}}$ must be written as $\frac{1+\frac{1}{x}}{\frac{1}{y}}$.

• Eliminate the complex fractions.

1. $\frac{\frac{2}{3}}{\frac{5}{6}}$

$$\boxed{\left(\frac{\frac{2}{3}}{\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{4}{5}}$$

2. $\frac{1+\frac{2}{3}}{1+\frac{5}{6}}$

$$\boxed{\left(\frac{1+\frac{2}{3}}{1+\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{6+4}{6+5}=\frac{10}{11}}$$

3. $\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}$

$$\boxed{\left(\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}\right)\left(\frac{12}{12}\right)=\frac{9+20}{24-2}=\frac{29}{22}}$$

4. $\frac{1+\frac{1}{2}x^{-1}}{1+\frac{1}{3}x^{-1}}$

$$\boxed{\left(\frac{1+\frac{1}{2x}}{1+\frac{1}{3x}}\right)\left(\frac{6x}{6x}\right)=\frac{6x+3}{6x+2}}$$

5. $\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}$

$$\boxed{\left(\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}\right)\left(\frac{4x^2}{4x^2}\right)=\frac{4x^3-2x}{4x^4+1}}$$

6. $\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}$

$$\boxed{\left(\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}\right)\left(\frac{15}{15}\right)=\frac{6x^{5/3}}{25}}$$

7. $\frac{x^{-3}+x}{x^{-2}+1}$

$$\boxed{\left(\frac{\frac{1}{x^3}+x}{\frac{1}{x^2}+1}\right)\left(\frac{x^3}{x^3}\right)=\frac{1+x^4}{x+x^3}}$$

8. $\frac{\frac{1}{2}(2x+5)^{-2/3}}{\frac{-2}{3}}$

$$\boxed{\left(\frac{\frac{1}{2}}{\frac{-2}{3}(2x+5)^{2/3}}\right)\frac{6}{6}=\frac{-3}{4(2x+5)^{2/3}}}$$

9. $\frac{(x-1)^{1/2}-\frac{x(x-1)^{-1/2}}{2}}{x-1}$

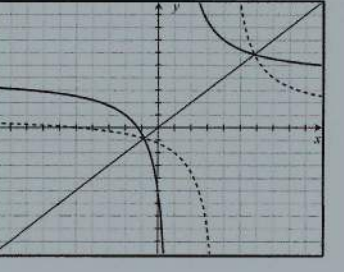
$$\boxed{\left(\frac{(x-1)^{1/2}-\frac{x}{2(x-1)^{1/2}}}{x-1}\right)\left[\frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}\right] \quad \frac{2(x-1)-x}{2(x-1)^{3/2}}=\frac{x-2}{2(x-1)^{3/2}}}$$

L. Inverses

No topic in math confuses students more than inverses. If a function is a rule that maps x to y, an **inverse** is a rule that brings y back to the original x. If a point (x, y) is a point on a function f, then the point (y, x) is on the inverse function f^{-1}. Students mistakenly believe that since $x^{-1} = \frac{1}{x}$, then $f^{-1} = \frac{1}{f}$. This is decidedly incorrect. If a function is given in equation form, to find the inverse, replace all occurrences of x with y and all occurrences of y with x. If possible, then solve for y. Using the "horizontal-line test" on the original function f will quickly determine whether or not f^{-1} is also a function. By definition, $f\left(f^{-1}(x)\right) = x$. The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

1. Find the inverse to $y = \dfrac{4x+5}{x-1}$ and show graphically that its inverse is a function.

$$\boxed{\text{Inverse: } x = \frac{4y+5}{y-1} \Rightarrow xy - x = 4y + 5 \Rightarrow y = \frac{x+5}{x-4}}$$



Note that the function is drawn in bold and the inverse as dashed. The function and its inverse is symmetrical to the line $y = x$. The inverse is a function for two reasons: a) it passes the vertical line text or b) the function passes the horizontal line test.

2. Find the inverse to the following functions and show graphically that its inverse is a function.

a. $y = 4x - 3$

Inverse: $x = 4y - 3$

$y = \dfrac{x+3}{4}$ (function)

b. $y = x^2 + 1$

Inverse: $x = y^2 + 1$

$y = \pm\sqrt{x-1}$ (not a function)

c. $y = x^2 + 4x + 4$

Inverse: $x = y^2 + 4y + 4$

$x = (y+2)^2 \Rightarrow \pm\sqrt{x} = y + 2$

$y = -2 \pm \sqrt{x}$ (not a function)

3. Find the inverse to the following functions and show that $f\left(f^{-1}(x)\right) = x$

a. $f(x) = 7x + 4$

Inverse: $x = 7y + 4$

$y = f^{-1}(x) = \dfrac{x-4}{7}$

$f\left(\dfrac{x-4}{7}\right) = 7\left(\dfrac{x-4}{7}\right) + 4 = x$

b. $f(x) = \dfrac{1}{x-1}$

Inverse: $x = \dfrac{1}{y-1} \Rightarrow xy - x = 1$

$y = f^{-1}(x) = \dfrac{x+1}{x}$

$f\left(\dfrac{x+1}{x}\right) = \left(\dfrac{1}{\frac{x+1}{x} - 1}\right)\left(\dfrac{x}{x}\right)$

$= \dfrac{x}{x+1-x} = x$

c. $f(x) = x^3 - 1$

Inverse: $x = y^3 - 1$

$y = f^{-1}(x) = \sqrt[3]{x+1}$

$f\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x$

4. Without finding the inverse, find the domain and range of the inverse to $f(x) = \sqrt{x+2} + 3$.

Function: Domain: $[-2, \infty)$, Range: $[3, \infty)$ Inverse: Domain: $[3, \infty)$, Range: $[-2, \infty)$

M. Adding Fractions and Solving Fractional Equations

There are two major problem types with fractions: Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one* … in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD*. When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1. a. Combine: $\frac{x}{3}-\frac{x}{4}$

LCD: 12 $\quad \frac{x}{3}\left(\frac{4}{4}\right)-\frac{x}{4}\left(\frac{3}{3}\right)$

$\frac{4x-3x}{12}=\frac{x}{12}$

b. Solve: $\frac{x}{3}-\frac{x}{4}=12$

$12\left(\frac{x}{3}\right)-12\left(\frac{x}{4}\right)=12(12)$

$4x-3x=144 \Rightarrow x=144$

$x=144: \frac{144}{3}-\frac{144}{4}=48-36=12$

2. a. Combine $x+\frac{6}{x}$

LCD: x $\quad x\left(\frac{x}{x}\right)+\frac{6}{x}$

$\frac{x^2+6}{x}$

b. Solve: $x+\frac{6}{x}=5$

$x(x)+x\left(\frac{6}{x}\right)=5x$

$x^2+6=5x \Rightarrow x^2-5x+6=0$

$(x-2)(x-3)=0 \Rightarrow x=2,\ x=3$

$x=2: 2+\frac{6}{2}=2+3=5 \quad x=3: 3+\frac{6}{3}=3+2=5$

3 . a. Combine: $\frac{12}{x+2}-\frac{4}{x}$

LCD: $x(x+2)$ $\quad \left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right)-\frac{4}{x}\left(\frac{x+2}{x+2}\right)$

$\frac{12x-4x-8}{x(x+2)}$

$\frac{8x-8}{x(x+2)}$

b. Solve $\frac{12}{x+2}-\frac{4}{x}=1$

$\frac{12}{x+2}(x)(x+2)-\frac{4}{x}(x)(x+2)=1(x)(x+2)$

$12x-4x-8=x^2+2x \Rightarrow x^2-6x+8=0$

$(x-2)(x-4)=0 \Rightarrow x=2,4$

$x=2: \frac{12}{4}-\frac{4}{2}=3-2=1 \quad x=4: \frac{12}{6}-\frac{4}{4}=2-1=1$

4. a. $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}$

LCD: $2(x-3)^2$

$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right)-\frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$

$\frac{x^2-3x-6}{2(x-3)^2}$

b. Solve $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}=\frac{x-2}{3x-9}$

$\left[\frac{x}{2(x-3)}-\frac{3}{(x-3)^2}=\frac{x-2}{3(x-3)}\right]6(x-3)^2$

$3x(x-3)-18=2(x-3)(x-2)$

$3x^2-9x-18=2x^2-10x+12$

$x^2+x-30=0 \Rightarrow (x+6)(x-5)=0 \Rightarrow x=-6,5$

$x=-6: \frac{-6}{-18}-\frac{3}{81}=\frac{-8}{-27} \quad x=5: \frac{5}{4}-\frac{3}{4}=\frac{3}{6}$

N. Solving Absolute Value Equations

Absolute value equations crop up in calculus, especially in BC calculus. The definition of the absolute value function is a piecewise function: $f(x)=|x|=\begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$. So, to solve an absolute value equation, split the absolute value equation into two equations, one with a positive parentheses and the other with a negative parentheses and solve each equation. It is possible that this procedure can lead to incorrect solutions so solutions must be checked.

- Solve the following equations.

1. $|x-1|=3$

$x-1=3$	$-(x-1)=3$
$x=4$	$-x+1=3$
	$x=-2$

2. $|3x+2|=9$

$3x+2=9$	$-(3x+2)=9$
$3x=7$	$-3x-2=9$
$x=\frac{7}{3}$	$3x=-11$
	$x=\frac{-11}{3}$

3. $|2x-1|-x=5$

$2x-1-x=5$	$-(2x-1)-x=5$
$x=6$	$-3x=4$
	$x=\frac{-4}{3}$

4. $|x+5|+5=0$

$x+5+5=0$	$-(x+5)+5=0$
$x=-10$	$-x-5+5=0$
	$x=0$

Both answers are invalid. It is impossible to add 5 to an absolute value and get 0.

5. $|x^2-x|=2$

$(x^2-x)=2$	$-(x^2-x)=2$
$x^2-x-2=0$	$-x^2+x=2$
$(x-2)(x+1)=0$	$0=x^2+x+2$
$x=2, x=-1$	No real solution

Both solutions check

6. $|x-10|=x^2-10x$

$x-10=x^2-10x$	$-(x-10)=x^2-10x$
$x^2-11x+10=0$	$-x+10=x^2-10x$
$(x-1)(x-10)=0$	$x^2-9x-10=0$
$x=1, x=10$	$(x-10)(x+1)=0$
	$x=10, x=-1$

Of the three solutions, only $x=-1$ and $x=10$ are valid.

7. $|x|+|2x-2|=8$

$x+2x-2=8$	$-x+2x-2=8$	$x-(2x-2)=8$	$-x-(2x-2)=8$
$3x=10$	$x=10$	$-x=6$	$-3x=6$
$x=\frac{10}{3}$		$x=-6$	$x=-2$

Of the four solutions, only $x=\frac{10}{3}$ and $x=-2$ are valid

O. Solving Inequalities

You may think that solving inequalities are just a matter of replacing the equal sign with an inequality sign. In reality, they can be more difficult and are fraught with dangers. And in calculus, inequalities show up more frequently than solving equations. Solving inequalities are a simple matter if they are based on linear equations. They are solved exactly like linear equations, remembering that if you multiply or divide both sides by a negative number, the direction of the inequality sign must be reversed.

However, if the inequality is more complex than a linear function, it is advised to bring all terms to one side. Pretend for a moment it is an equation and solve. Then create a number line which determines whether the transformed inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the zeroes are included or not.

If the inequality involves an absolute value, create two equations, replacing the absolute value with a positive parentheses and a negative parentheses and the inequality sign with an equal sign. Solve each, placing each solution on your number line. Then determine which intervals satisfy the original inequality.

If the inequality involves a rational function, set both numerator and denominator equal to zero, which will give you the values you need for your number line. Determine whether the inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the endpoints are included or not.

- Solve the following inequalities.

1. $2x-8\le 6x+2$

$-10\le 4x$ or $-4x\le 10$

$\dfrac{-5}{2}\le x$ or $x\ge\dfrac{-5}{2}$

2. $1-\dfrac{3x}{2}>x-5$

$2-3x>2x-10$

$12>5x\Rightarrow x<\dfrac{12}{5}$

3. $-7\le 6x-1<11$

$-6\le 6x\le 12$

$-1\le x\le 2$

4. $|2x-1|\le x+4$

$|2x-1|-x-4\le 0$

$2x-1-x-4=0$ $\quad -2x+1-x-4=0$

$x=5$ $\quad x=-1$

++++++0---------0+++++++

-1 5

So $-1\le x\le 5$ or $[-1,5]$

5. $x^2-3x>18$

$x^2-3x-18>0\Rightarrow(x+3)(x-6)>0$

For $(x+3)(x-6)=0$, $x=-3, x=6$

++++++0-------0+++++++

-3 6

So $x<-3$ or $x>6$ or $(-\infty,-3)\cup(6,\infty)$

6. $\dfrac{2x-7}{x-5}\le 1$

$\dfrac{2x-7}{x-5}-1=0\Rightarrow\dfrac{2x-7}{x-5}-\dfrac{x-5}{x-5}<0\Rightarrow\dfrac{x-2}{x-5}<0$

++++++0-------∞+++++++

2 5

So $2\le x<5$ or $[2,5)$

7. Find the domain of $\sqrt{32-2x^2}$

$2(4+x)(4-x)\ge 0$

------0+++++++0-------

-4 4

So $-4\le x\le -4$ or $[-4,4]$

P. Exponential Functions and Logarithms

Calculus spends a great deal of time on exponential functions in the form of b^x. Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If $y = b^x$ then $x = \log_b y$. So when you are trying to find the value of $\log_2 32$, state that $\log_2 32 = x$ and $2^x = 32$ and therefore $x = 5$.

If the base of a log statement is not specified, it is defined to be 10. When we asked for log 100, we are solving the equation: $10^x = 100$ and $x = 2$. The function $y = \log x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$. In calculus, we primarily use logs with base *e*, which are called natural logs (ln). So finding ln 5 is the same as solving the equation $e^x = 5$. Students should know that the value of $e = 2.71828\ldots$

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

i. $\log a + \log b = \log(a \cdot b)$ ii. $\log a - \log b = \log\left(\frac{a}{b}\right)$ iii. $\log a^b = b \log a$

1. Find a. $\log_4 8$

$$\boxed{\begin{aligned} &\log_4 8 = x \\ &4^x = 8 \Rightarrow 2^{2x} = 2^3 \\ &x = \frac{3}{2} \end{aligned}}$$

b. $\ln\sqrt{e}$

$$\boxed{\begin{aligned} &\ln\sqrt{e} = x \\ &e^x = e^{1/2} \\ &x = \frac{1}{2} \end{aligned}}$$

c. $10^{\log 4}$

$$\boxed{\begin{aligned} &\log 4 = x \\ &10^x = 4 \text{ so } 10^{\log 4} = 4 \\ &\text{10 to a power and log are inverses} \end{aligned}}$$

d. $\log 2 + \log 50$

$$\boxed{\begin{aligned} &\log(2 \cdot 50) = \log 100 \\ &2 \end{aligned}}$$

e. $\log_4 192 - \log_4 3$

$$\boxed{\begin{aligned} &\log_4\left(\frac{192}{3}\right) \\ &\log_4 64 = 3 \end{aligned}}$$

f. $\ln\sqrt[5]{e^3}$

$$\boxed{\ln e^{3/5} = \frac{3}{5}\ln e = \frac{3}{5}}$$

2. Solve a. $\log_9\left(x^2 - x + 3\right) = \frac{1}{2}$

$$\boxed{\begin{aligned} &x^2 - x + 3 = 9^{1/2} \\ &x(x-1) = 0 \\ &x = 0, x = 1 \end{aligned}}$$

b. $\log_{36} x + \log_{36}(x-1) = \frac{1}{2}$

$$\boxed{\begin{aligned} &\log_{36} x(x-1) = \frac{1}{2} \\ &x(x-1) = 36^{1/2} = 6 \\ &x^2 - x - 6 = 0 \\ &(x-3)(x+2) = 0 \\ &\text{Only } x = 3 \text{ is in the domain} \end{aligned}}$$

c. $\ln x - \ln(x-1) = 1$

$$\boxed{\begin{aligned} &\ln\left(\frac{x}{x-1}\right) = 1 \\ &\frac{x}{x-1} = e \Rightarrow x = ex - e \\ &x = \frac{e}{e-1} \end{aligned}}$$

d. $5^x = 20$

$$\boxed{\begin{aligned} &\log(5^x) = \log 20 \\ &x \log 5 = \log 20 \\ &x = \frac{\log 20}{\log 5} \text{ or } x = \frac{\ln 20}{\ln 5} \end{aligned}}$$

e. $e^{-2x} = 5$

$$\boxed{\begin{aligned} &\ln e^{-2x} = \ln 5 \\ &-2x = \ln 5 \Rightarrow x = \frac{-\ln 5}{2} \end{aligned}}$$

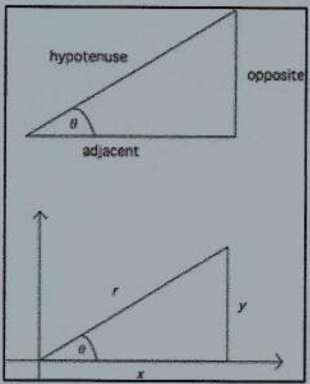
f. $2^x = 3^{x-1}$

$$\boxed{\begin{aligned} &\log(2^x) = \log(3^{x-1}) \\ &x \log 2 = (x-1)\log 3 \\ &x \log 2 = x \log 3 - \log 3 \Rightarrow x = \frac{\log 3}{\log 3 - \log 2} \end{aligned}}$$

Q. Right Angle Trigonometry

Trigonometry is an integral part of AP calculus. Students must know the basic trig function definitions in terms of opposite, adjacent and hypotenuse as well as the definitions if the angle is in standard position.

Given a right triangle with one of the angles named θ, and the sides of the triangle relative to θ named opposite (y), adjacent (x), and hypotenuse (r) we define the 6 trig functions to be:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyptotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyptotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

The Pythagorean theorem ties these variables together: $x^2 + y^2 = r^2$. Students should recognize right triangles with integer sides: 3-4-5, 5-12-13, 8-15-17, 7-24-25. Also any multiples of these sides are also sides of a right triangle. Since r is the largest side of a right triangle, it can be shown that the range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$, the range of $\csc\theta$ and $\sec\theta$ is $(-\infty,-1]\cup[1,\infty)$ and the range of $\tan\theta$ and $\cot\theta$ is $(-\infty,\infty)$.

Also vital to master is the signs of the trig functions in the four quadrants. A good way to remember this is A – S – T – C where All trig functions are positive in the 1st quadrant, Sin is positive in the 2nd quadrant, Tan is positive in the 3rd quadrant and Cos is positive in the 4th quadrant.

1. Let be a point on the terminal side of θ. Find the 6 trig functions of θ. (Answers need not be rationalized).

a) $P(-8,6)$

$x=-8, y=6, r=10$

$\sin\theta = \frac{3}{5}$ $\qquad \csc\theta = \frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\qquad \sec\theta = -\frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ $\qquad \cot\theta = -\frac{4}{3}$

b. $P(1,3)$

$x=1,\ y=3,\ r=\sqrt{10}$

$\sin\theta = \frac{3}{\sqrt{10}}$ $\qquad \csc\theta = \frac{\sqrt{10}}{3}$

$\cos\theta = \frac{1}{\sqrt{10}}$ $\qquad \sec\theta = \sqrt{10}$

$\tan\theta = 3$ $\qquad \cot\theta = \frac{1}{3}$

c. $P(-\sqrt{10},-\sqrt{6})$

$x=-\sqrt{10},\ y=-\sqrt{6},\ r=4$

$\sin\theta = -\frac{\sqrt{6}}{4}$ $\qquad \csc\theta = -\frac{4}{\sqrt{6}}$

$\cos\theta = -\frac{\sqrt{10}}{4}$ $\qquad \sec\theta = -\frac{4}{\sqrt{10}}$

$\tan\theta = \sqrt{\frac{3}{5}}$ $\qquad \cot\theta = \sqrt{\frac{5}{3}}$

2. If $\cos\theta = \frac{2}{3}$, θ in quadrant IV, find $\sin\theta$ and $\tan\theta$

$x=2,\ r=3,\ y=-\sqrt{5}$

$\sin\theta = -\frac{\sqrt{5}}{3}, \tan\theta = -\frac{\sqrt{5}}{2}$

3. If $\sec\theta = \sqrt{3}$ find $\sin\theta$ and $\tan\theta$

θ is in quadrant I or IV

$x=1,\ y=\pm\sqrt{2},\ r=\sqrt{3}$

$\sin\theta = \pm\sqrt{\frac{2}{3}},\ \tan\theta = \pm\sqrt{2}$

4. Is $3\cos\theta + 4 = 2$ possible?

$3\cos\theta = -2$

$\cos\theta = -\frac{2}{3}$ which is possible.

R. Special Angles

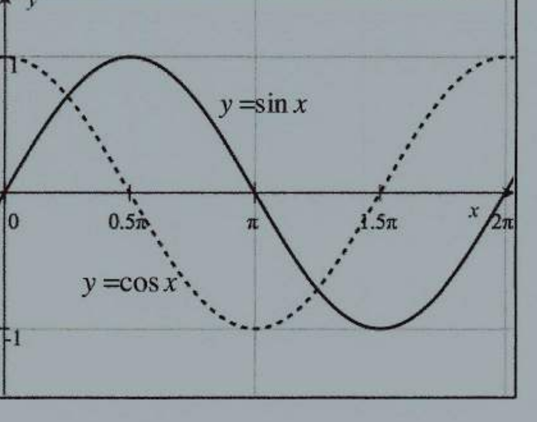
Students must be able to find trig functions of quadrant angles $(0,\ 90°, 180°, 270°)$ and special angles, those based on the $30°-60°-90°$ and $45°-45°-90°$ triangles.

First, for most calculus problems, angles are given and found in radians. Students must know how to convert degrees to radians and vice-versa. The relationship is 2π radians $= 360°$ or π radians $= 180°$. Angles are assumed to be in radians so when an angle of $\frac{\pi}{3}$ is given, it is in radians. However, a student should be able to picture this angle as $\frac{180°}{3} = 60°$. It may be easier to think of angles in degrees than radians, but realize that unless specified, angle measurement must be written in radians. For instance, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

The trig functions of **quadrant angles** $\left(0,\ 90°, 180°, 270° \text{ or } 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right)$ can quickly be found. Choose a point along the angle and realize that r is the distance from the origin to that point and always positive. Then use the definitions of the trig functions.

θ	point	x	y	r	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$(1,0)$	1	0	1	0	1	0	does not exist	1	does not exist
$\frac{\pi}{2}$ or $90°$	$(0,1)$	0	1	1	1	0	does not exist	1	does not exist	0
π or $180°$	$(-1,0)$	-1	0	1	0	-1	0	does not exist	-1	does not exist
$\frac{3\pi}{2}$ or $270°$	$(0,-1)$	0	-1	1	-1	0	Does not exist	-1	does not exist	0

If you picture the graphs of $y = \sin x$ and $y = \cos x$ as shown to the right, you need not memorize the table. You must know these graphs backwards and forwards.



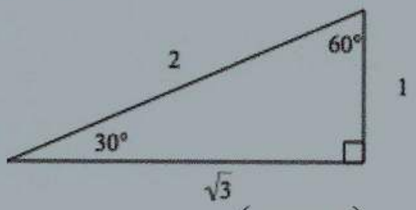
• Without looking at the table, find the value of

a. $5\cos 180° - 4\sin 270°$

$$\boxed{\begin{array}{l}5(-1)-4(-1)\\ -5+4=-1\end{array}}$$

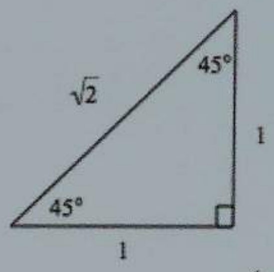
b. $\left(\dfrac{8\sin\frac{\pi}{2} - 6\tan\pi}{5\sec\pi - \csc\frac{3\pi}{2}}\right)^2$ $\quad\boxed{\left[\dfrac{8(1)-6(0)}{5(-1)-(-1)}\right]^2 = \left(\dfrac{8}{-4}\right)^2 = 4}$

Because over half of the AP exam does not use a calculator, you must be able to determine trig functions of **special angles**. You must know the relationship of sides in both $30°-60°-90° \left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ and $45°-45°-90° \left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangles.



In a $30°-60°-90° \left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ triangle,

the ratio of sides is $1-\sqrt{3}-2$.



In a $45°-45°-90° \left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangle,

the ratio of sides is $1-1-\sqrt{2}$.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30° \left(\text{or } \frac{\pi}{6}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45° \left(\text{or } \frac{\pi}{4}\right)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60° \left(\text{or } \frac{\pi}{3}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Special angles are any multiple of $30° \left(\frac{\pi}{6}\right)$ or $45°\left(\frac{\pi}{2}\right)$. To find trig functions of any of these angles, draw them and find the **reference angle** (the angle created with the x-axis). Although most problems in calculus will use radians, you might think easier using degrees. This will create one of the triangles above and trig functions can be found, remembering to include the sign based on the quadrant of the angle. Finally, if an angle is outside the range of $0°$ to $360°(0 \text{ to } 2\pi)$, you can always add or subtract $360°(2\pi)$ to find trig functions of that angle. These angles are called **co-terminal angles**. It should be pointed out that $390° \neq 30°$ but $\sin 390° = \sin 30°$.

• Find the exact value of the following

a. $4\sin 120° - 8\cos 570°$

Subtract 360° from 570°
$4\sin 120° - 8\cos 210°$
120° is in quadrant II with reference angle 60°.
210° is in quadrant III with reference angle 30°.
$4\left(\frac{\sqrt{3}}{2}\right)-8\left(\frac{-\sqrt{3}}{2}\right)=6\sqrt{3}$

b. $\left(2\cos\pi - 5\tan\frac{7\pi}{4}\right)^2$

$(2\cos 180° - 5\tan 315°)^2$
180° is a quadrant angle
315° is in quadrant III with reference angle 45°
$[2(-1)-5(-1)]^2 = 9$

S. Trigonometric Identities

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier. The following chart gives the major trig identities that you should know. To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

<u>Fundamental Trig Identities</u>

$$\csc x = \frac{1}{\sin x},\quad \sec x = \frac{1}{\cos x},\quad \cot x = \frac{1}{\tan x},\quad \tan x = \frac{\sin x}{\cos x},\quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1,\quad 1+\tan^2 x = \sec^2 x,\quad 1+\cot^2 x = \csc^2 x$$

<u>Sum Identities</u>

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

<u>Double Angle Identities</u>

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x = 1-2\sin^2 x = 2\cos^2 x - 1$$

- Verify the following identities.

1. $\left(\tan^2 x+1\right)\left(\cos^2-1\right) = -\tan^2 x$

$$\left(\sec^2 x\right)\left(-\sin^2 x\right)$$
$$\left(\frac{1}{\cos^2 x}\right)\left(-\sin^2 x\right)$$
$$-\tan^2 x$$

2. $\sec x - \cos x = \sin x\tan x$

$$\frac{1}{\cos x} - \cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$$
$$\sin x\left(\frac{\sin x}{\cos x}\right) = \sin x\tan x$$

3. $\dfrac{\cot^2 x}{1+\csc x} = \dfrac{1-\sin x}{\sin x}$

$$\left(\frac{\dfrac{\cos^2}{\sin^2 x}}{1+\dfrac{1}{\sin x}}\right)\frac{\sin^2 x}{\sin^2 x} = \frac{\cos^2 x}{\sin^2 x+\sin x}$$
$$\frac{1-\sin^2 x}{\sin x(1+\sin x)} = \frac{(1+\sin x)(1-\sin x)}{\sin x(1+\sin x)}$$
$$\frac{1-\sin x}{\sin x}$$

4. $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$

$$\left(\frac{1+\sin x}{\cos x}\right)\left(\frac{1+\sin x}{1+\sin x}\right) + \left(\frac{\cos x}{1+\sin x}\right)\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1+2\sin x+\sin^2 + \cos^2 x}{\cos x(1+\sin x)}$$
$$\frac{1+2\sin x+1}{\cos x(1+\sin x)} = \frac{2+2\sin x}{\cos x(1+\sin x)}$$
$$\frac{2(1+\sin x)}{\cos x(1+\sin x)} = 2\sec x$$

5. $\cos^4 2x - \sin^4 2x = \cos 4x$

$$\left(\cos^2 2x+\sin^2 2x\right)\left(\cos^2 2x-\sin^2 2x\right)$$
$$1\left[\cos 2(2x)\right]$$
$$\cos 4x$$

6. $\sin(3\pi - x) = \sin x$

$$\sin 3\pi\cos x - \cos 3\pi\sin x$$
$$0(\cos x) - (-1)\sin x = \sin x$$

T. Solving Trig Equations and Inequalities

Trig equations are equations using trig functions. Typically they have many (or infinite) number of solutions so usually they are solved within a specific domain. Without calculators, answers are either quadrant angles or special angles, and again, they must be expressed in radians.

For trig inequalities, set both numerator and denominator equal to zero and solve. Make a sign chart with all these values included and examine the sign of the expression in the intervals. Basic knowledge of the sine and cosine curve is invaluable from section R is invaluable.

• Solve for x on $[0,2\pi)$

1. $x\cos x = 3\cos x$

Do not divide by $\cos x$ as you will lose solutions

$\cos x(x-3)=0$

$\cos x = 0 \qquad x-3=0$

$x=\frac{\pi}{2},\frac{3\pi}{2} \qquad x=3$

You must work in radians.

Saying $x=90°$ makes no sense.

2. $\tan x+\sin^2 x = 2-\cos^2 x$

$\tan x+\sin^2 x+\cos^2 x=2$

$\tan x+1=2$

$\tan x=1$

$x=\frac{\pi}{4},\frac{5\pi}{4}$

Two answers as tangent is positive in quadrants I and III.

3. $3\tan^2 x-1=0$

$3\tan^2 x=1$

$\tan^2 x=\frac{1}{3}$

$\tan x=\pm\sqrt{\frac{1}{3}}=\pm\frac{\sqrt{3}}{3}$

$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

4. $3\cos x=2\sin^2 x$

$3\cos x=2\left(1-\cos^2 x\right)$

$2\cos^2 x+3\cos x-2=0$

$(2\cos x-1)(\cos x+2)=0$

$2\cos x=1 \qquad \cos x=-2$

$\cos x=\frac{1}{2} \qquad$ No solution

$x=\frac{\pi}{3},\frac{5\pi}{3}$

7. Solve for x on $[0,2\pi)$: $\frac{2\cos x+1}{\sin^2 x}>0$

$2\cos x=-1 \Rightarrow \cos x=\frac{-1}{2} \Rightarrow x=\frac{2\pi}{3},\frac{4\pi}{3}$

$\sin^2 x=0 \Rightarrow x=0,\pi$

0	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	2π
+++++++	0 -------	∞ -------	0 +++++++	

Answer: $\left[0,\frac{2\pi}{3}\right)\cup\left(\frac{4\pi}{3},2\pi\right)$

U. Graphical Solutions to Equations and Inequalities

You have a shiny new calculator. So when are we going to use it? So far, no mention has been made of it. Yet, the calculator is a tool that is required in the AP calculus exam. For about 25% of the exam, a calculator is permitted. So it is vital you know how to use it.

There are several settings on the calculator you should make. First, so you don't get into rounding difficulties, it is suggested that you set your calculator to three decimal places. That is a standard in AP calculus so it is best to get into the habit. To do so, press MODE and on the 2nd line, take it off FLOAT and change it to 3. And second, set your calculator to radian mode from the MODE screen. There may be times you might want to work in degrees but it is best to work in radians.

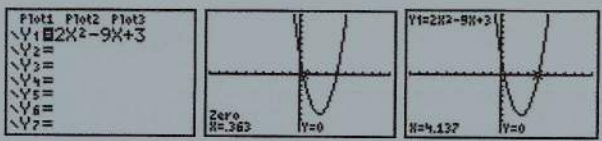
You must know how to graph functions. The best way to graph a function is to input the function using the Y= key. Set your XMIN and XMAX using the WINDOW key. Once you do that, you can see the graph's total behavior by pressing ZOOM 0. To evaluate a function at a specific value of x, the easiest way to do so is to press these keys: VARS → 1:Function 1 1:Y1 (and input your x-value.

Other than basic calculations, and taking trig functions of angles, there are three calculator features you need to know: evaluating functions at values of x and finding zeros of functions, which we know is finding where the function crosses the x-axis. The other is finding the point of intersection of two graphs. Both of these features are found on the TI-84+ calculator in the CALC menu 2ND TRACE. They are 1:value, 2: zero, and 5: intersect.

Solving equations using the calculator is accomplished by setting the equation equal to zero, graphing the function, and using the ZERO feature. To use it, press 2ND TRACE ZERO . You will be prompted to type in a number to the left of the approximate zero, to the right of the approximate zero, and a guess (for which you can press ENTER). You will then see the zero (the solution) on the screen.

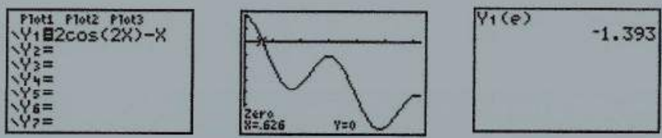
• Solve these equations graphically.

1. $2x^2-9x+3=0$



This equation can be solved with the quadratic formula. $x=\dfrac{9\pm\sqrt{81-24}}{4}=\dfrac{9\pm\sqrt{57}}{4}$

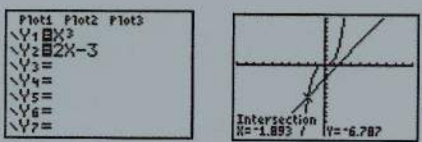
2. $2\cos 2x-x=0$ on $[0,2\pi)$ and find $2\cos(2e)-e$.



If this were the inequality $2\cos 2x-x>0$, the answer would be $[0,0.626)$.

3. Find the x-coordinate of the intersection of $y=x^3$ and $y=2x-3$

You can use the intersection feature.



Or set them equal to each other: $x^3=2x-3$ or $x^3-2x+3=0$